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Some properties of yarn

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1. Introduction

A yarn, of nonnegative cross-section, lies along a t-axis with $-\infty < t < \infty$. It is composed of fibres, which are cylinders with their axes parallel to the t-axis. The head of a fibre is its left endpoint on the t-axis, the tail its right endpoint. Fibre heads are distributed along the t-axis according to a Poisson-process with parameter $\lambda(t)$, where $\lambda(t)$ is Lebesgue-integrable on any finite interval and $0 < \lambda(t) \leq \lambda$ for all t and some finite λ. We define such a process in the following way: Regarding the t-axis as composed of intervals, of which the j^{th} interval is $(j, j+1]$ (open on the left, closed on the right), where j runs through the integers, the distribution of the number $\underline{r}_j$ of heads falling in the j^{th} interval is given by

$$(1.1) \qquad P\{\underline{r}_j = r\} = e^{-\lambda_j} \frac{\lambda_j^r}{r!} \quad \text{for } r \geq 0$$

with

$$(1.2) \qquad \lambda_j \stackrel{\text{def}}{=} \int_j^{j+1} \lambda(t)\, dt .$$

The head of the k^{th} fibre $(k = 1, 2, \ldots, r;\ \underline{r}_j = r \geq 1)$ having its head in the j^{th} interval lies in $\underline{t}_{j,k}$, where under the condition $\underline{r}_j = r$ the $\underline{t}_{j,1}, \ldots, \underline{t}_{j,r}$ are independently distributed random variables with the common distribution function $K_j(t)$, with

$$(1.3) \qquad K_j(t) \stackrel{\text{def}}{=} P\{\underline{t}_{j,1} \leq t\} = \begin{cases} 0 & \text{for } t \leq j , \\ \lambda_j^{-1} \int_j^t \lambda(u)\, du & \text{for } j \leq t \leq j+1 , \\ 1 & \text{for } j+1 \leq t . \end{cases}$$

The $\underline{t}_{j,k}$ for different intervals on the t-axis are mutually independent.

The length $\underline{x}\ (\geq 0)$ and cross-section $\underline{y}\ (\geq 0)$ of a fibre have a simultaneous distribution function $H(x,y) = P\{\underline{x} \leq x, \underline{y} \leq y\}$. The vectors $(\underline{x}, \underline{y})$ belonging to different fibres are mutually independent and independent of the location of the fibre on the t-axis. A realization of the yarn is a stepfunction, which can be taken continuous from the right. The length of the k^{th} fibre having its head in the j^{th} interval is $\underline{x}_{j,k}$, its cross-section $\underline{y}_{j,k}$.

Object of this paper is the study of random variables like $\underline{c}(t_o)$ and $\underline{v}(t_o, t_o + h)$, which denote the cross-section of the yarn at t_o and the volume of yarn in the interval $(t_o, t_o + h]$ respectively (with t_o arbitrary and $h > 0$).

2. Terpstra's approach

In this section we assume that $H(x_o, \infty) = 1$ for some $x_o < \infty$, i.e. the fibres have length $\leq x_o$ with probability 1. Furthermore we take $\lambda(t) = \lambda > 0$, i.e. we consider a stationary Poisson - process (this restriction need not be made).

In order to derive $\mathcal{E}\, e^{i\tau \underline{c}(t_o)}$, we first compute

$$(2.1) \qquad \varphi_{\underline{c}}(\tau, t_o; s, l) \stackrel{\text{def}}{=} \mathcal{E}\{e^{i\tau \underline{c}(t_o)} \mid \underline{n}(s, s+l) = 1\},$$

where $\underline{n}(s, s+l)$ denotes the number of heads of fibres in the interval $(s, s+l]$ and $s < t_o - x_o < t_o < s+l$.

If $\underline{t}$ is the coordinate of the head of the one fibre falling in $(s, s+l]$, then it is known that $\underline{t}$ is uniformly distributed in this interval. If the fibre has length $\underline{x} = x$, with probability $\frac{x}{l}$ the head lies in $(t_o - x, t_o]$. Hence

$$(2.2) \qquad \varphi_{\underline{c}}(\tau, t_o; s, l) = \int_0^\infty \int_0^\infty \left[e^{i\tau y} \frac{x}{l} + e^0 \left(1 - \frac{x}{l}\right) \right] dH(x,y) =$$

$$= 1 - \frac{1}{l} \int_0^\infty \int_0^\infty x(1 - e^{i\tau y})\, dH(x,y)\, .$$

As

$$(2.3) \qquad P\{\underline{n}(s, s+l) = n\} = e^{-\lambda l} \frac{(\lambda l)^n}{n!} \qquad \text{for } n = 0, 1, 2, \ldots$$

and $\underline{c}(t_o)$ is the sum of the independent contributions to the cross - section of all fibres with heads in $(s, s+l]$, we have (the characteristic function of a sum of independent random variables being equal to the product of the characteristic functions of the individual terms)

$$(2.4) \qquad \varphi_{\underline{c}}(\tau, t_o) \stackrel{\text{def}}{=} \mathcal{E}\, e^{i\tau \underline{c}(t_o)} =$$

$$= \sum_{n=0}^{\infty} \{\varphi_{\underline{c}}(\tau, t_o; s, l)\}^n e^{-\lambda l} \frac{(\lambda l)^n}{n!} =$$

$$= \exp - \lambda l \left\{1 - \left(1 - \frac{1}{l} \int_0^\infty \int_0^\infty x(1 - e^{i\tau y})\, dH(x,y)\right)\right\} =$$

$$= \exp - \lambda \int_0^\infty \int_0^\infty x(1 - e^{i\tau y})\, dH(x,y)\, .$$

As was to be expected, $\varphi_{\underline{c}}(\tau, t_o)$ does not depend on either s or l. We may conjecture, that (2.4) will also hold if $H(x, \infty) < 1$ for all $x < \infty$. (Some restriction is needed: we must have $\mathcal{E}\, \underline{x} < \infty$, cf. Breny (1957) and section 4.)

We remark that from (2.4) it is evident that $\underline{c}(t_0)$ has a compound Poisson - distribution, i.e.

(2.5) $$\underline{c}(t_0) = \sum_{j=1}^{\underline{n}} \underline{b}_j ,$$

where

(2.6) $$P\{\underline{n} = n\} = e^{-\lambda \mathcal{E} \underline{x}} \frac{(\lambda \mathcal{E} \underline{x})^n}{n!} \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

and, under the condition $\underline{n} = n$, the $\underline{b}_j$ are independently distributed, each with distribution function

(2.7) $$G(y) \overset{\text{def}}{=} \frac{\int_0^\infty x \, d_x H(x,y)}{\mathcal{E} \underline{x}} .$$

Analogously to the derivation of (2.4) one can show that

(2.8) $$\varphi_{\underline{v}}(\tau, t_0, h) \overset{\text{def}}{=} \mathcal{E} e^{i\tau \underline{v}(t_0, t_0+h)} =$$

$$= \exp - \lambda [h + \mathcal{E}\underline{x} + \int_{y=0}^{\infty} \int_{x=0}^{h} \{ \frac{2}{iy\tau} (1 - e^{i\tau xy}) - (h - x) e^{i\tau xy} \} dH(x,y) +$$

$$+ \int_{y=0}^{\infty} \int_{x=h}^{\infty} \frac{2}{iy\tau} (1 - e^{i\tau hy}) - (x - h) e^{i\tau hy} \, dH(x,y)] ,$$

always assuming a finite maximum length x_0 for the fibres.

Both (2.4) and (2.8) are due to Terpstra. These results are known for $\underline{y} = 1$ with probability 1 (cf. Spencer - Smith and Todd (1941), Martindale (1945), Breny (1952) and Breny (1953), Olerup (1952)). Related considerations as to method of derivation are to be found in Fortet (1951).

3. Van Dantzig's method

In this section we assume that $\underline{y} \equiv 1$ (or $\underline{y} = 1$ with probability 1).

In Breny (1957) relation (2.8) (with $\underline{y} \equiv 1$) is obtained by applying a limiting procedure to (in our notation) $\mathcal{E} \exp i \sum_{j=0}^{n} \tau_j \underline{n}(t_j)$, expressed as a complicated sum of double integrals. As Van Dantzig pointed out to Breny (cf. Breny (1957), page 33) one might proceed in the following manner. Let A denote a Lebesgue - measurable set in the space $\Omega = \{(t, x) \mid -\infty < t < \infty, 0 \leq x < \infty\}$ and $\underline{m}(A)$ the number of fibres for which $(\underline{t}_{j,k}, \underline{x}_{j,k}) \in A$ is satisfied. Then, if $A \cap B = 0$, $\underline{m}(A)$ and $\underline{m}(B)$

are independent stochastic variables having a Poisson - distribution, the parameter of $\underline{m}(A)$ being

$$(3.1) \qquad \mathcal{E}\,\underline{m}(A) = \iint_A \lambda(t)\,dt\,dF(x)\,,$$

where $F(x) \overset{\text{def}}{=} P\{\underline{x} \leq x\}$ is the distribution function of the length of a fibre. For a general class of real functions $\xi(t,x)$ one can define

$$(3.2) \qquad \int_0^{\infty}\int_{-\infty}^{\infty} \xi(t,x)\,d\underline{m}\,,$$

where $\underline{m}$ is a stochastic measure on Ω, with $\underline{m}(A)$ as described, for every L - measurable set A. Because of the independence of the $\underline{m}(A)$ for disjoint sets, we shall have something like (use Riemann - sums)

$$(3.3) \qquad \mathcal{E}\exp i\tau \int_0^{\infty}\int_{-\infty}^{\infty} \xi(t,x)\,d\underline{m} \approx \prod_{\nu=0}^{\infty}\prod_{\mu=-\infty}^{\infty} \mathcal{E}\, e^{i\tau\xi(t^*_\mu,x^*_\nu)\underline{m}(\Delta_{\mu\nu})} \approx$$

$$\approx \prod_{\nu=0}^{\infty}\prod_{\mu=-\infty}^{\infty}\left\{(1-\lambda(t)\,\Delta t\,\Delta F(x))e^0 + \lambda(t)\,\Delta t\,\Delta F(x)e^{i\tau\xi(t^*_\mu,x^*_\nu)}\right\} \approx$$

$$\approx \prod_{\nu=0}^{\infty}\prod_{\mu=-\infty}^{\infty} \exp - \lambda(t)\{1 - i\tau\xi(t^*_\mu,x^*_\nu)\}\Delta t\,\Delta F(x) \approx$$

$$\approx \exp - \int_0^{\infty}\int_{-\infty}^{\infty} \lambda(t)\{1 - i\tau\xi(t,x)\}\,dt\,dF(x)$$

or

$$(3.4) \qquad \mathcal{E}\exp i\tau \int_0^{\infty}\int_{-\infty}^{\infty} \xi(t,x)\,d\underline{m} = \exp - \int_0^{\infty}\int_{-\infty}^{\infty} \lambda(t)\{1 - i\tau\xi(t,x)\}\,dt\,dF(x)\,.$$

Substitution of

$$(3.5) \qquad \xi(t,x) = \begin{cases} 1 & \text{for } t \leq t_0,\ t+x > t_0 \\ 0 & \text{otherwise} \end{cases}\,,$$

now leads to (2.4), because

$$(3.6) \qquad \underline{c}(t_0) = \int_0^{\infty}\int_{t_0 - x < t \leq t_0} d\underline{m}\,.$$

In the same way

$$(3.7) \qquad \xi(t,x) = \{(t-t_0)\,\iota(t_0-t) - (t-t_0-h)\,\iota(t_0+h-t)\} +$$
$$- \{(x+t-t_0)\,\iota(t_0-x-t) - (x+t-t_0-h)\,\iota(t_0+h-x-t)\}$$

leads to (2.8).

Stochastic set functions have been discussed by Prékopa (cf. Prékopa (1956, 1957)).

4. Main formula

The k^{th} fibre from the j^{th} interval contributes to the cross-section at point t of the axis an amount

$$(4.1) \qquad \underline{c}_{j,k}(t) \overset{\text{def}}{=} \underline{y}_{j,k}\{\iota(\underline{t}_{j,k}+\underline{x}_{j,k}-t) - \iota(\underline{t}_{j,k}-t)\},$$

where $\iota(x) = 1$ for $x \geq 0$ and $\iota(x) = 0$ for $x < 0$. Hence we find for the total cross-section at point t [1)]

$$(4.2) \qquad \underline{c}(t) \overset{\text{def}}{=} \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\underline{r}_j} \underline{y}_{j,k}\{\iota(\underline{t}_{j,k}+\underline{x}_{j,k}-t) - \iota(\underline{t}_{j,k}-t)\}.$$

We shall study the random variable

$$(4.3) \qquad \underline{u}_T \overset{\text{def}}{=} \int_{-\infty}^{\infty} \underline{c}(t)\, dT(t)$$

under the condition $\mathcal{E}\,\underline{x} < \infty$. This variable is (as will be shown) well-defined for any real-valued function $T(t)$, which is of bounded variation in a closed interval $[t_1, t_2]$ (with $-\infty < t_1 < t_2 < \infty$), constant for $t \leq t_1$, as well as constant for $t \geq t_2$ and finally continuous from the right for all t. We assume (this is no restriction) that t_1 is a negative integer and t_2 a positive integer.

With probability 1 $\underline{c}(t)$ is a stepfunction with a finite number of finite steps in the interval $[t_1, t_2]$, given $\mathcal{E}\,\underline{x} < \infty$ (Breny (1957)). As inside $[t_1, t_2]$ only a finite number of finite steps originate, it is sufficient to prove that with probability 1 only a finite number of fibres cover the point t_1. Thus we have to prove

$$(4.4) \qquad P\{\sum_{j=-\infty}^{t_1-1} \sum_{k=1}^{\underline{r}_j} \iota(\underline{t}_{j,k}+\underline{x}_{j,k}-t_1) < \infty\} = 1$$

or

1) We take $\sum_{k=1}^{\underline{r}_j} \ldots = 0$ for $\underline{r}_j = 0$, here and later.

(4.5) $$P\{\max_{1 \le k \le \underline{r}_j} (\underline{t}_{j,k} + \underline{x}_{j,k} - t_1) \ge 0 \quad \text{for infinitely many } j\} = 0 .$$

To prove the last relation, it is sufficient to show (by one of the Borel - Cantelli lemmas) that for the (independent) events

(4.6) $$A_j \overset{\text{def}}{=} \{\max_{1 \le k \le \underline{r}_j} (\underline{t}_{j,k} + \underline{x}_{j,k} - t_1) \ge 0\}$$

we have

(4.7) $$\sum_{j=-\infty}^{t_1-1} P\{A_j\} < \infty .$$

Now, if $F(x) \overset{\text{def}}{=} P\{\underline{x} \le x\}$,

(4.8) $$\begin{aligned} P\{A_j\} &= P\{\max_{1 \le k \le \underline{r}_j} (\underline{t}_{j,k} + \underline{x}_{j,k} - t_1) \ge 0\} = \\ &= \sum_{r=1}^{\infty} e^{-\lambda_j} \frac{\lambda_j^r}{r!} \{1 - [\int_j^{j+1} F(t_1 - t)\, dK_j(t)]^r\} = \\ &= 1 - \exp - \lambda_j \{1 - \int_j^{j+1} F(t_1 - t)\, dK_j(t)\} \le \\ &\le \lambda_j \{1 - \int_j^{j+1} F(t_1 - t)\, dK_j(t)\} \le \lambda \{1 - F(t_1 - j - 1)\} , \end{aligned}$$

as $1 - e^{-x} \le x$ for $x \ge 0$ and $\lambda_j \le \lambda$ for each j . Hence, because $\sum_{k=1}^{\infty} \{1 - F(k)\} \le \int_0^{\infty} x\, dF(x) = \mathcal{E}\underline{x} < \infty$, condition (4.7) is satisfied.

For a stochastic stepfunction $\underline{c}(t)$ which has, with probability 1 , a finite number of finite steps in the interval $[t_1, t_2]$, the integral in (4.3) is defined with probability 1 as the Lebesgue - Stieltjes integral of the realization $c(t)$ and satisfies

(4.9) $$\begin{aligned} \underline{u}_T &= \int_{-\infty}^{\infty} \underline{c}(t)\, dT(t) = \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\underline{r}_j} \underline{y}_{j,k} \int_{-\infty}^{\infty} \{\iota(\underline{t}_{j,k} + \underline{x}_{j,k} - t) - \iota(\underline{t}_{j,k} - t)\}\, dT(t) = \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\underline{r}_j} \underline{y}_{j,k} \{T(\underline{t}_{j,k} + \underline{x}_{j,k}) - T(\underline{t}_{j,k})\} , \end{aligned}$$

where the double series has only a finite number of nontrivial terms (i.e. terms unequal 0) with probability 1 .

Lemma. If $\underline{a}_n$, $n = 1, 2, \ldots$ are mutually independent real-valued stochastic variables, such that the events $A_n = \{\underline{a}_n \neq 0\}$ satisfy $P\{A_n$ occurs for infinitely many $n\} = 0$, we have: $\underline{s} \overset{\text{def}}{=} \sum_{n=1}^{\infty} \underline{a}_n$ is a stochastic variable for which (for all real τ)

$$(4.10) \qquad \mathcal{E}\, e^{i\tau \underline{s}} = \prod_{n=1}^{\infty} \mathcal{E}\, e^{i\tau \underline{a}_n} .$$

Proof: $\underline{s}$ is a well-defined stochastic variable, for it is with probability 1 the sum of a finite number of independent random variables unequal 0. As we have

$$(4.11) \qquad \{A_n \text{ occurs for infinitely many } n\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

and therefore

$$(4.12) \qquad \lim_{n\to\infty} P\{\bigcup_{m=n}^{\infty} A_m\} = 0 ,$$

to each $\varepsilon > 0$ there exists an $N = N(\varepsilon)$, for which $P\{\bigcup_{m=N+1}^{\infty} A_m\} < \varepsilon$ or $P\{\bigcap_{m=N+1}^{\infty} \bar{A}_m\} > 1 - \varepsilon$ (if $\bar{A}_m$ denotes the complement of the set A_m). But then

$$(4.13) \qquad \left| \mathcal{E} \prod_{n=1}^{\infty} e^{i\tau \underline{a}_n} - \mathcal{E} \prod_{n=1}^{N} e^{i\tau \underline{a}_n} \right| \leq \left| \mathcal{E} \prod_{n=1}^{N} e^{i\tau \underline{a}_n} \right| \cdot \left| \mathcal{E} \prod_{n=N+1}^{\infty} e^{i\tau \underline{a}_n} - 1 \right| \leq$$

$$\leq \mathcal{E}\left\{ \left| \prod_{n=N+1}^{\infty} e^{i\tau \underline{a}_n} - 1 \right| \,\Big|\, \underline{a}_n \neq 0 \text{ for at least one } n \geq N+1 \right\} \cdot \varepsilon \leq 2\varepsilon ,$$

and hence

$$(4.14) \qquad \lim_{N\to\infty} \prod_{n=1}^{N} \mathcal{E}\, e^{i\tau \underline{a}_n} = \mathcal{E} \prod_{n=1}^{\infty} e^{i\tau \underline{a}_n} ,$$

which proves the lemma.

The random variables

$$(4.15) \qquad \underline{z}_j \overset{\text{def}}{=} \prod_{k=1}^{\underline{r}_j} \exp i\tau \underline{y}_{j+k} \{T(\underline{t}_{j,k} + \underline{x}_{j,k}) - T(\underline{t}_{j,k})\}$$

are nontrivial, i.e. not equal to 1 with probability 1, only for $j \leq \underline{t}_2 - 1$. Hence we find from (4.9) by applying the lemma to

$$(4.16) \qquad \underline{a}_n \overset{\text{def}}{=} \underline{z}_{t_2-n} ,$$

that for all real τ

(4.17) $$\mathcal{E} \exp i\tau \underline{u}_T =$$

$$= \mathcal{E} \prod_{j=-\infty}^{\infty} \prod_{k=1}^{\underline{r}_j} \exp i\tau \underline{y}_{j,k} \{T(\underline{t}_{j,k} + \underline{x}_{j,k}) - T(\underline{t}_{j,k})\} =$$

$$= \prod_{j=-\infty}^{\infty} \mathcal{E} \prod_{k=1}^{\underline{r}_j} \exp i\tau \underline{y}_{j,k} \{T(\underline{t}_{j,k} + \underline{x}_{j,k}) - T(\underline{t}_{j,k})\} .$$

Further we have by (1,1), (1.2) and (1.3), using Fubini's theorem to prove the second equality,

(4.18) $$\mathcal{E} \prod_{k=1}^{\underline{r}_j} \exp i\tau \underline{y}_{j,k} \{T(\underline{t}_{j,k} + \underline{x}_{j,k}) - T(\underline{t}_{j,k})\} =$$

$$= \mathcal{E} \sum_{r=0}^{\infty} \left\{ \prod_{k=1}^{r} \exp i\tau \underline{y}_{j,k} [T(\underline{t}_{j,k} + \underline{x}_{j,k}) - T(\underline{t}_{j,k})] \,\Big|\, \underline{r}_j = r \right\} P\{\underline{r}_j = r\} =$$

$$= \sum_{r=0}^{\infty} \{ \int_0^{\infty} \int_0^{\infty} \int_j^{j+1} \exp i\tau y [T(t+x) - T(t)] \, dK_j(t) \, dH(x,y)\}^r P\{\underline{r}_j = r\} =$$

$$= \exp \{- \lambda_j (1 - \int_0^{\infty} \int_0^{\infty} \int_j^{j+1} \exp i\tau y [T(t+x) - T(t)] \, dK_j(t) \, dH(x,y))\} =$$

$$= \exp - \int_0^{\infty} \int_0^{\infty} \int_j^{j+1} \lambda(t) [1 - \exp i\tau y \{T(t+x) - T(t)\}] \, dt \, dH(x,y) .$$

Because $|1 - \exp i\tau y \{T(t+x) - T(t)\}| \leq 2$ and $T(t+x) - T(t) = 0$ for $t \leq t_1 - x$ and $t \geq t_2$, we have

(4.19) $$\int_{-\infty}^{\infty} |1 - \exp i\tau y \{T(t+x) - T(t)\}| \, dt \leq 2\{x + (t_2 - t_1)\}$$

and so, because

(4.20) $$\int_0^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} |1 - \exp i\tau y \{T(t+x) - T(t)\}| \, dt \, dH(x,y) \leq 2\{\mathcal{E}\underline{x} + (t_2 - t_1)\} < \infty ,$$

we are allowed to apply Fubini's theorem once again. Combining (4.17) and (4.18) we find our <u>main formula</u>

(4.21) $$\mathcal{E} \exp i\tau \underline{u}_T =$$

$$= \exp - \sum_{j=-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \int_j^{j+1} \lambda(t) [1 - \exp i\tau y \{T(t+x) - T(t)\}] \, dt \, dH(x,y) =$$

$$= \exp - \int_0^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \lambda(t) [1 - \exp i\tau y \{T(t+x) - T(t)\}] \, dt \, dH(x,y) .$$

The particular cases

$$T(t) = \iota(t - t_o), \tag{4.22}$$

$$T(t) = (t - t_o)\,\iota(t - t_o) - (t - t_o - h)\,\iota(t - t_o - h), \tag{4.23}$$

show, that both (2.4) and (2.8) are satisfied for $\mathcal{E}\,\underline{x} < \infty$. In fact a generalization of these results (with $\lambda(t)$ instead of λ) has been obtained.

Relation (4.21) may be used to obtain interesting formulae for other stochastic variables besides $\underline{c}(t_o)$ and $\underline{v}(t_o, t_o + h)$.

Literature

H. Breny (1952), Quelques considérations sur la théorie statistique des faisceaux de fibres, Rapport S 96, Département de Statistique du Centre Mathématique, Amsterdam.

H. Breny (1953), The calculation of the variance - length curve from the length distribution of fibres, I and II, Jl. Text. Inst., 44, P 1 - P 9 and P 10 - P 14.

H. Breny (1957), Recherches sur la théorie statistique des faisceaux de fibres, Thèse, Mémoires de la Soc. Roy. des Sciences de Liège, XVII - II, 7 - 131.

R. Fortet (1951), Random functions from a Poisson process, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Univ. of California Press, Berkeley and Los Angeles.

J.G. Martindale (1945), A new method of measuring the irregularity of yarns with some observations on the origin of irregularities in worsted slivers and yarns, Jl. Text. Inst., 36, T 35 - T 47.

H. Olerup (1952), Calculation of the variance - length curve for an ideal sliver, Jl. Text. Inst., 43, P 290 - P 293.

A. Prékopa (1956, 1957), On stochastic set functions I, II, III, Acta Mathematica Hung., 7 (1956), 215 - 263, 8 (1957), 337 - 374, 375 - 400.

J.L. Spencer - Smith and H.A.C. Todd (1941), A time series met with in textile research, Jl. Roy. Stat. Soc. Supp., 7, 131 - 145.